Chapter Seven

Probabilities
Of Living and Dying

7/1
Life (or Mortality**) Table**

7/1/1 The Life (or Mortality) Table

It follows from the idea of a group of persons attaining age X and being gradually reduced in numbers, until they are all dead, by the operation of mortality in such a way that the rates of mortality at successive ages form a smooth series is a purely theoretical conception. It is, nevertheless, a very useful conception, which forms the basis of the theory of life contingencies and has been shown by long use to be suitable for solving most practical problems in life assurance and similar work. The fundamental function of the mortality table is a function known as l_x.

This hypothetical function corresponds to the empirical sequence n_x, which has been discussed above. It is a positive non-increasing function representing the number of lives who are expected to survive to age x out of l_x lives that attain age x, this being the youngest age for which l_x is tabulated. For theoretical purposes it is convenient to assume that l_x is a continuous function, having values for all x and not just integers and with unique finite differential coefficients to any required order at all points.

Just as $\Delta n_x / n_x$ – is the observed rate of mortality at age x, so- $\Delta l_x / l_x$ is the rate of mortality

at age x assumed in the mortality table. The mortality assumption, on which the table is based, may take the form of an assumed rate of mortality for each integral age x.

Since, given such a series of rates and a suitable arbitrary value of l_x, a column of l_x can be constructed for integral values of x. In fact, the l_x column in most mortality tables is constructed in this way. In this event, although l_x will be defined for integral ages only, we can assume that it exists for all values of x, integral and fractional. Sometimes l_x is defined by a mathematical formula, in which event we can obtain the rates of mortality by calculating numerical values of l_x for integral ages. In this case, l_x will be defined for all values of x.

As described, l_x represents the number of lives who, according to the mortality table, **are expected to survive to age x of l_x lives that attain age *x*.**

Since the mortality table can start at any age, l_{x+t} may be regarded as representing the number of lives who are expected to survive to age x+t out of l_x lives aged x; both x and x+t not necessarily being integers.

Although in the definition of the mortality table, nothing was said about the various characteristics of the individual hypothetical lives. The l_x lives in the mortality table, like the n_x

persons in the illustrative mortality investigation, are not necessarily to be regarded as all exactly identical from the point of view of the various characteristics, which influence mortality.

l_{x+t}/l_x is a function, which expresses the mortality assumption in the form of the expected proportions surviving to each age and hence it gives the probability relations as applicable to random samples.

The ratio l_{x+t}/l_x is the probability of survival to age x+1 in respect of an individual life aged x taken at random from an indefinitely large number of lives who are assumed to experience, as a whole, the mortality on which the table is based. It will be seen that $1-l_{x+t}/l_x$ is analogous to the 'cumulative distribution function' of probability theory. In dealing with problems in life contingencies, whenever we refer to a life aged x we imply unless a specific statement is made to the contrary, that a life taken at random is intended.

If a random sample of *m* lives aged *x* is taken, the expected number surviving to age x+1 is ml_{x+t}/ l_x but the actual number is subject to random variations. For sufficiently large values of *m* these random variations are small relative to the expected number. Many of the functions developed in this book are similarly subject to random variations. Practical work usually

concerns group of lives, which may be regarded as random samples.

These random variations are usually small in relation to the errors involved in the choice of a mortality table, to variations in mortality from year to year and to errors arising from deviations of actual rates of interest and expenses from those assumed. Accordingly they are usually ignored in practice, but the fact that they exist should not be overlooked and some examples are described in Chapter 11.

A possible table l_x, which starts at age 0 and ceases at age 100 is shown in Table 1.1.

We would then have the probability (rounded to four decimal places**) of a member of the group at birth, taken at random, living to exact age 1 as 0.9929 and to exact age 2 as 0.9912. If we wished to find the probability of a member aged exactly 1 living to exact age 2 we would divide l_2 by l_1 that is 9,911,725 by 9,929,200 to give 0.9982.**

It is important to remember that the figures in the l_x column have no meaning individually and are only meaningful when related to each other – thus the probability of a person living from exact age 20 to exact age 30 is l_{30}/l_{20} = 9,480,358/ 9,664,994 = .9809. While the table should strictly be interpreted only in this way this ratio is sometimes said verbally to be the

'number alive at age 30' divided by the 'number alive at age 20'.

Because the figures in the table have no meaning except when divided by each other. All the figures in a mortality table could be multiplied by a factor and no change would be made in the mortality represented. Thus Table 1.2a and b would apart from the rounding of certain of the figures represent the same mortality, the first being the previous table multiplied by 1/100, and the second by ½.

Table 1.1

العمر س Age x	ح س l_x عدد الأحياء
0	10,000.000
1	9,929.200
2	9,911.725
3	9,896.659
:	
10	9,805.870
:	
20	9,664.994
:	
30	9,480.358
:	
40	9,241.359
:	
50	8,762.306
:	

Age x	l_x
60	7,698.698
:	
70	5,592.012
:	
80	2626.372
:	
90	468.174
:	
99	6.415
100	0

While the table shown commences at age 0 this is not necessary and, for example, a mortality table intended for use in life assurance offices to show the mortality of annuitants might commence at age 40. Normally the lx figure for the lowest age is taken as some convenient round number such as 1,000,000,999,999 or 500,000- this is called the radix of the table. The first age at which the value of l_x becomes negligible is called the limiting age and is denoted by ∞, so that l_∞ = 0- in the table above ∞ =100. Some mortality tables are assumed to conform to a mathematical formula so that the l_x column converges asymptotically to zero and never actually becomes 0. However, for practical purposes, a limiting age such as 100,105 or 110 is assumed for all mortality tables.

Table 1.2a

X	l_x
0	100.000
1	99.292
:	
10	98.059
:	
20	96.650
:	

Table 1.2b

x	lx
0	5,000,000
1	4,964,600
:	
10	4,902,935
:	
20	4,832,497
:	

7/1/2 International Actuarial Notations:

The existing international actuarial notation was founded on the "Key to the Notation" given in the *Institute of Actuaries Text Book, Part II, Life Contingencies* by George King (1887), and was adopted by the Second International Actuarial Congress, London, 1898 (*Transactions*, pp. 618–640) with minor revisions approved by the Third International Congress, Paris, 1900 (*Transactions*, pp. 622–651). Further revisions were discussed during 1937–1939, and were introduced by the Institute and the Faculty in 1949 (*J.I.A.*, 75, 121 and *T.F.A.*, 19, 89). These revisions were finally adopted internationally at the Fourteenth International Actuarial Congress, Madrid, 1954 (*Bulletin of the Permanent Committee of the International Congress of Actuaries* (1949))

The general principles on which the system is based are as follows:

To each fundamental symbolic letter are attached signs and letters each having its own signification.

The lower space to the left is reserved for signs indicating the conditions relative to the duration of the operations and to their position with regard to time.

The lower space to the right is reserved for signs indicating the conditions relative to ages and the order of succession of the events.

The upper space to the right is reserved for signs indicating the periodicity of the events.

The upper space to the left is free, and in it can be placed signs corresponding to other notions.

In what follows these two conventions are used:

A letter enclosed in brackets, thus (x), denotes 'a person aged x'.

A letter or number enclosed in a right angle, thus $\overline{n|}$ or $\overline{15|}$, denotes a term-certain of years.

FUNDAMENTAL SYMBOLIC LETTERS

Interest

i = the effective rate of interest, namely, the total interest earned on 1 in a year on the assumption that the actual interest (if receivable otherwise than yearly) is invested forthwith as it becomes due on the same terms as the original principal.

$v = (1+i)^{-1}$ = the present value of 1 due a year hence.

$d = 1 - v$ = the discount on 1 due a year hence.

$\delta = \log_e(1+i) = -\log_e(1-d)$ = the force of interest or the force of discount.

Mortality Tables

l = number living.

d = number dying.

p = probability of living.

q = probability of dying.

μ = force of mortality.

m = central death rate.

a = present value of an annuity.

s = amount of an annuity.

e = expectation of life.

A = present value of an assurance.

E = present value of an endowment.

P = premium per annum. } P generally refers to net premiums, π to special

π = premium per annum. } premiums.

V = policy value.

W = paid-up policy.

The methods of using the foregoing principal letters and their precise meaning when added to by suffixes, etc., follow.

Interest

$i^{(m)} = m\{(1+i)^{1/m} - 1\}$ = the nominal rate of interest, convertible m times a year.

$a_{\overline{n}|} = v + v^2 + \ldots + v^n$ = the value of an annuity-certain of 1 per annum for n years, the payments being made at the end of each year.

$\ddot{a}_{\overline{n}|} = 1 + v + v^2 + \ldots + v^{n-1}$ = the value of a similar annuity, the payments being made at the beginning of each year.

$s_{\overline{n|}} = 1 + (1+i) + (1+i)^2 + \ldots + (1+i)^{n-1}$ = the amount of an annuity-certain of 1 per annum for n years, the payments being made at the end of each year.

$\ddot{s}_{\overline{n|}} = (1+i) + (1+i)^2 + \ldots + (1+i)^n$ = the amount of a similar annuity, the payments being made at the beginning of each year.

The diaeresis or trema above the letters a and s is used as a symbol of acceleration of payments.

Mortality Tables

The ages of the lives involved are denoted by letters placed as suffixes in the lower space to the right. Thus:

l_x = the number of persons who attain age x according to the mortality table.

$d_x = l_x - l_{x+1}$ = the number of persons who die between ages x and $x+1$ according to the mortality table.

p_x = the probability that (x) will live 1 year.

q_x = the probability that (x) will die within 1 year.

$\mu_x = -\dfrac{1}{l_x}\dfrac{dl_x}{dx}$ = the force of mortality at age x.

m_x = the central death-rate for the year of age x to $x+1$ $= d_x \Big/ \int_0^1 l_{x+t}\,dt$.

e_x = the curtate 'expectation of life' (or average after-lifetime) of (x).

In the following it is always to be understood (unless otherwise expressed) that the annual payment of an annuity is 1, that the sum assured in any case is 1, and that the symbols indicate the present values:

a_x = an annuity, first payment at the end of a year, to continue during the life of (x).

$\ddot{a}_x = 1 + a_x$ = an 'annuity-due' to continue during the life of (x), the first payment to be made at once.

A_x = an assurance payable at the end of the year of death of (x).

Note. $e_x = a_x$ at rate of interest $i = 0$.

A letter or number at the lower left corner of the principal symbol denotes the number of years involved in the probablity or benefit in question. Thus:

${}_{n}p_{x}$ = the probability that (x) will live n years.
${}_{n}q_{x}$ = the probability that (x) will die within n years.

Note. When $n=1$ it is customary to omit it, as shown on page 98, provided no ambiguity is introduced.

${}_{n}E_{x}=v^{n}{}_{n}p_{x}$ = the value of an endowment on (x) payable at the end of n years if (x) be then alive.

If the letter or number comes before a perpendicular bar it shows that a period of deferment is meant. Thus:

${}_{n|}q_{x}$ = the probability that (x) will die in a year, deferred n years; that is, that he will die in the $(n+1)$th year.
${}_{n|}a_{x}$ = an annuity on (x) deferred n years; that is, that the first payment is to be made at the end of $(n+1)$ years.
${}_{n|t}a_{x}$ = an intercepted or deferred temporary annuity on (x) deferred n years and, after that, to run for t years.

A letter or number in brackets at the upper right corner of the principal symbol shows the number of intervals into which the year is to be divided. Thus:

$a_{x}^{(m)}$ = an annuity on (x) payable by m instalments of $1/m$ each throughout the year, the first payment being one of $1/m$ at the end of the first $1/m$th of a year.
$\ddot{a}_{x}^{(m)}$ = a similar annuity but the first payment of $1/m$ is to be made at once, so that
$\ddot{a}_{x}^{(m)}=1/m+a_{x}^{(m)}$.
$A_{x}^{(m)}$ = an assurance payable at the end of that fraction $1/m$ of a year in which (x) dies.

If $m\rightarrow\infty$, then instead of writing (∞) a bar is placed over the principal symbol. Thus:

$\bar{a}$ = a continuous or momently annuity.
$\bar{A}$ = an assurance payable at the moment of death.

A small circle placed over the principal symbol shows that the benefit is to be complete. Thus:

$\mathring{a}$ = a complete annuity.
$\mathring{e}$ = the complete expectation of life.

Note. Some consider that $\bar{e}$ would be as appropriate as $\mathring{e}$. As $e_x = a_x$ at rate of interest $i=0$, so also the complete expectation of life $= \bar{a}_x$ at rate of interest $i=0$.

When more than one life is involved the following rules are observed:

If there are two or more letters or numbers in a suffix without any distinguishing mark, joint lives are intended. Thus:

$$l_{xy} = l_x \times l_y, \quad d_{xy} = l_{xy} - l_{x+1:y+1}.$$

Note. When, for the sake of distinctness, it is desired to separate letters or numbers in a suffix, a colon is placed between them. A colon is used instead of a point or comma to avoid confusion with decimals when numbers are involved.

a_{xyz} = an annuity, first payment at the end of a year, to continue during the joint lives of (x), (y) and (z).
A_{xyz} = an assurance payable at the end of the year of the failure of the joint lives (x), (y) and (z).

In place of a life a term-certain may be involved. Thus:

$a_{x:\overline{n|}}$ = an annuity to continue during the joint duration of the life of (x) and a term of n years certain; that is, a temporary annuity for n years on the life of (x).
$A_{x:\overline{n|}}$ = an assurance payable at the end of the year of death of (x) if he die within n years, or at the end of n years if (x) be then alive; that is, an endowment assurance for n years.

If a perpendicular bar separates the letters in the suffix, then the status after the bar is to follow the status before the bar. Thus:

$a_{y|x}$ = a reversionary annuity, that is, an annuity on the life of (x) after the death of (y).
$A_{z|xy}$ = an assurance payable on the failure of the joint lives (x) and (y) provided both these lives survive (z).

If a horizontal bar appears above the suffix then survivors of the lives, and not joint lives, are intended. The number of survivors can be denoted by a letter or number over the right end of the bar. If that letter, say r, is not distinguished by any mark, then the meaning is *at least* r survivors; but if it is enclosed in square brackets, $[r]$, then the meaning is *exactly* r survivors. If no letter or number appears over the bar, then unity is supposed and the meaning is *at least one* survivor. Thus:

$a_{\overline{xyz}}$ = an annuity payable so long as at least one of the three lives (x), (y) and (z) is alive.

$a_{\overline{xyz}}^{\;2}$ = an annuity payable so long as at least two of the three lives (x), (y) and (z) are alive.

$p_{\overline{xyz}}^{[2]}$ = probability that exactly two of the three lives (x), (y) and (z) will survive a year.

${}_nq_{\overline{xy}}$ = probability that the survivor of the two lives (x) and (y) will die within n years $= {}_nq_x \times {}_nq_y$.

${}_nA_{\overline{xy}}$ = an assurance payable at the end of the year of death of the survivor of the lives (x) and (y) provided the death occurs within n years.

When numerals are placed above or below the letters of the suffix, they designate the order in which the lives are to fail. The numeral placed *over* the suffix points out the life whose failure will finally determine the event; and the numerals placed *under* the suffix indicate the order in which the other lives involved are to fail. Thus:

$A^{\;1}_{xy}$ = an assurance payable at the end of the year of death of (x) if he dies first of the two lives (x) and (y).

$A^{\;2}_{xyz}$ = an assurance payable at the end of the year of death of (x) if he dies second of the three lives (x), (y) and (z).

$A^{\;2}_{x\underset{1}{y}z}$ = an assurance payable at the end of the year of death of (x) if he dies second of the three lives, (y) having died first.

$A_{\overline{xy}:\underset{3}{z}}$ = an assurance payable at the end of the year of death of the survivor of (x) and (y) if he dies before (z).

$A^{\;1}_{x:\overline{n}|}$ = an assurance payable at the end of the year of death of (x) if he dies within a term of n years.

$a_{\overline{y\underset{1}{z}}|x}$ or $a^{\;2}_{\overline{yz}|x}$ = an annuity to (x) after the failure of the survivor of (y) and (z), provided (z) fails before (y).

Note. Sometimes to make quite clear that a joint-life status is involved a symbol ⌐ is placed above the lives included. Thus $A^{\;1}_{\overline{xy}:\overline{n}|}$ = a joint-life temporary assurance on (x) and (y).

In the case of reversionary annuities, distinction has sometimes to be made between those where the times of year at which payments are to take place are determined at the outset and those where the times depend on the failure of the preceding status. Thus:

$a_{y|x}$ = annuity to (x), first payment at the end of the year of the death of (y) or, on the average, about 6 months after his death.

$\hat{a}_{y|x}$ = annuity to (x), first payment 1 year after the death of (y).

$\mathring{\hat{a}}_{y|x}$ = complete annuity to (x), first payment 1 year after the death of (y).

ANNUAL PREMIUMS

The symbol P with the appropriate suffix or suffixes is used in simple cases, where no misunderstanding can occur, to denote the annual premium for a benefit. Thus:

P_x = the annual premium for an assurance payable at the end of the year of death of (x).

$P_{x:\overline{n}|}$ = the annual premium for an endowment assurance on (x) payable after n years or at the end of the year of death of (x) if he die within n years.

P^{1}_{xy} = the annual premium for a contingent assurance payable at the end of the year of death of (x) if he die before (y).

In all these cases it is optional to use the symbol P in conjunction with the principal symbol denoting the benefit. Thus instead of $P_{x:\overline{n}|}$ we may write $P(A_{x:\overline{n}|})$. In the more complicated cases it is necessary to use the two symbols in this way. Suffixes, etc., showing the conditions of the benefit are to be attached to the principal letter, and those showing the condition of payment of the premium are to be attached to the subsidiary symbol P. Thus:

${}_nP(\bar{A}_x)$ = the annual premium payable for n years only for an assurance payable at the moment of the death of (x).

$P_{xy}(A_x)$ = the annual premium payable during the joint lives of (x) and (y) for an assurance payable at the end of the year of death of (x).

${}_nP({}_{n|}a_x)$ = the annual premium payable for n years only for an annuity on (x) deferred n years.

${}_tP^{(m)}(A_{x:\overline{n|}})$ = the annual premium payable for t years only, by m instalments throughout the year, for an endowment assurance for n years on (x) (see below as to $P^{(m)}$).

Notes. (1) As a general rule the symbol P could be used without the principal symbol in the case of assurances where the sum assured is payable at the end of the year of death, but if it is payable at other times, or if the benefit is an annuity, then the principal symbol should be used.

(2) $P_x^{(m)}$. A point which was not brought out when the international system was adopted is that there are two kinds of premiums payable m times a year, viz. those which cease on payment of the instalment immediately preceding death and those which continue to be payable to the end of the year of death. To distinguish the latter the m is sometimes enclosed in square brackets, thus $P^{[m]}$.

POLICY VALUES AND PAID-UP POLICIES

${}_tV_x$ = the value of an ordinary whole-life assurance on (x) which has been t years in force, the premium then just due being unpaid.

${}_tW_x$ = the paid-up policy the present value of which is ${}_tV_x$.

The symbols V and W may, in simple cases, be used alone, but in the more complicated cases it is necessary to insert the full symbol for the benefit thus:

$${}_tV^{(m)}(\bar{A}_{x:\overline{n|}}) \text{ (corresponding to } P^{(m)}(\bar{A}_{x:\overline{n|}})\text{)}, \quad {}_tV({}_{n|}a_x).$$

Note. As a general rule V or W can be used as the main symbol if the sum assured is payable at the end of the year of death and the premium is payable periodically throughout the duration of the assurance. If the premium is payable for a limited number of years, say n, the policy value after t years could be written ${}_tV[{}_nP(A)]$, or, if desired, ${}^n_tV(A)$.

In investigations where modified premiums and policy values are in question such modification may be denoted by adding accents to the symbols. Thus, when a premium other than the net premium (a valuation premium) is used in a valuation it may be denoted by P' and the corresponding policy value by V'. Similarly, the office (or commercial) premium may be denoted by P'' and the corresponding paid-up policy by W''.

7/2
Probabilities of Living and Dying-rate of Mortality

Table 1.3 is an extract of a mortality table, which will be used to exemplify the notation used in this section. This introduces the symbol d_x, which is the difference between successive lx figures. Thus

$$d_x = l_x - l_{x+1}, \text{ or } d_x = -\Delta l_x. \qquad (1.3.1)$$

In the convenient (but strictly incorrect**) interpretation of the mortality table mentioned in the previous section d_x represents the "number dying aged x last birthday".**

Table1.3

x	lx	dx
40	80,935	455
41	80,480	481
42	79,999	511
43	79,488	546
44	78,942	585
45	78,357	626
:	:	:
50	74,794	844

Probability of death and survival can be obtained directly from lx and d_x columns of the

mortality table. The symbol (x) is used to denote 'a person aged, x', or 'a life aged x'.

The probability that (x) will survive to age x+1, is denoted by p_x, so

$$p_x = \frac{l_{x+1}}{l_x} \qquad (1.3.2)$$

thus $$p_{40} = \frac{l_{41}}{l_{40}} = \frac{80.480}{80.935} = .9944.$$

The probability of a person age x dying during the year, the rate of mortality, is given the symbol q_x, so

$$q_x = \frac{d_x}{l_x} = \frac{l_x - l_{x+1}}{l_x} = \frac{-\Delta lx}{lx} \qquad (1.3.3)$$

thus $$q_{40} = \frac{d_{40}}{l_{40}} = \frac{455}{80,935} = .0056.$$

The probability of (x) living for a year plus the probability of (x) dying within a year must obviously be1, so

$$p_x + q_x = 1, \text{or } p_x = 1 - q_x, \text{ or } q_x = 1 - p_x. \qquad (1.3.4)$$

The probability that (x) will live for *n* years to age *x+n* is given the symbol ${}_np_x$;

$${}_np_x = \frac{l_{x+n}}{l_x} \qquad (1.3.5)$$

for example $${}_5p_{40} = \frac{78,357}{80,935} = .9681.$$

Another way of looking at ${}_np_x$ is to consider it as the product of successive probabilities of living each year

$${}_np_x = \frac{l_{x+1}}{l_x} \cdot \frac{l_{x+2}}{l_{x+1}} \cdot \frac{l_{x+3}}{l_{x+2}} \cdots \frac{l_{x+n}}{l_{x+n-1}}$$

$$= p_x \cdot p_{x+1} \cdot p_{x+2} \cdots p_{x+n-1}.$$

The probability that (x) will die within *n* years is given the symbol ${}_nq_x$;

$$ {}_nq_x = \frac{l_x - l_{x+n}}{l_x} = \frac{1}{l_x} \sum_{y=x}^{x+n-1} d_y = 1 - {}_np_x \qquad (1.3.6)$$

so $${}_5q_{40} = \frac{80{,}935 - 78{,}357}{80{,}935} \text{ or } \frac{455+481+511+546+585}{80935} \text{ or } 1 - .9681$$

all equalling .0319.

It can be seen that if no figure prefixes the symbol *p* or *q*, 1 is assumed, i.e. ${}_1p_x = p_x$; ${}_1q_x = q_x$.

The probability that (x) will live *m* years but die in the following *n* years, or that (x) will die between ages x+m and x+m+n is given the symbol.

$${}_{m|n}q_x = \frac{l_{x+m} - l_{x+m+n}}{l_x} = \frac{l_{x+m}}{l_x} \cdot \frac{l_{x+m} - l_{x+m+n}}{l_{x+m}} \qquad (1.3.7)$$

$$= {}_mp_x \cdot {}_nq_{x+m}.$$

Thus $${}_{3|2}q_{40} = \frac{l_{43} - l_{45}}{l_{40}} = \frac{79{,}488 - 78{,}357}{80{,}935} = \frac{1{,}131}{80{,}935}$$

$$= .0140.$$

When $n=1$ it may be omitted and the symbol becomes

$${}_{m|}q_x = \frac{d_{x+m}}{l_x} = {}_mp_x \cdot q_{x+m}$$

being the probability that (x) will die in a year, deferred *m* years;

thus $${}_{2|}q_{42} = \frac{d_{44}}{l_{42}} = \frac{585}{79{,}999} = .0073.$$

Mortality tables:

It will be seen that a mortality table is defined either by the l_x or q_x (or p_x) columns. If the q_x figures are given, the l_x column will be obtained by choosing a suitable radix (1_0) and successively obtaining the d_x and l_x figures:-

$d_0=l_0.q_0$
$l_1=l_0-d_0$
$d_1=l_1.q_1$ **and generally**
$d_x=l_x.q_x$
$l_{x+1}=l_x-d_x.$

The radix is arbitrary and is chosen so that a suitable degree of accuracy is obtained when the l_x figures are divided. Apart from this point the absolute value of an individual value of l_x is of no consequence. There is no reason why the value of l_x should not be recorded in decimals and this is sometimes done at the high ages of a table in order to obtain a smoother run of figures.

The examples in this book will normally use the tables in appendix III- The English Life Table No. 12- Males, the A1967-70 Table for Assured Lives and the A (55) tables for Annuitants.

The English Life Table No. 12-Males is constructed from the mortality rates experienced by the male population in England in the years 1960, 1961 and 1962. The A1967-70 table is

based on the experience within these years of lives assured of United Kingdom life assurance companies (and has the unusual radix of 34,489 decided so that value of the largest function calculated did not exceed the capacity of the computer**). The a (55) tables give the rates of mortality separately for males and females, based on the mortality experience of annuitants of United Kingdom life offices in 1946 to 1948, but projected into the future so that estimated rates were obtained thought to be applicable for annuities purchased in 1955.**

Mortality rates have been tending to decrease at most ages and in most countries of the world for more than a hundred years and published tables thus gradually cease to represent the experienced mortality. The English Life Tables are published at ten-year intervals, being based on the ten-yearly population census, while the assurance and annuitant tables are published at longer intervals. It might be thought that tables would become out of date quickly but is common practice to adjust published tables, the most usual method being to deduct (or perhaps add**) a certain number of years to the actual age before entering the table. One example of this is the common deduction of about four years from the age of a female policyholder in calculating the premiums for a life assurance policy, the published premium rates having been based on male mortality.**

Example 1.1:

Using the mortality table given in section 1.2, find the probability of a person aged 30,

(i) surviving to age40

(ii) dying before age40

(iii) dying between the ages of 60 and 80.

Solution:

(i) $_{10}p_{30} = \frac{l_{40}}{l_{30}} = \frac{9{,}241{,}359}{9{,}480{,}358} = 9748$

(ii) $_{10}q_{30} = \frac{l_{30} - l_{40}}{l_{30}} = \frac{238{,}999}{9{,}480{,}358} = .0252$ or,

$_{10}q_{30} = 1 - {}_{10}p_{30} = 1 - .9748 = .0252$

(iii) $_{30|20}q_{30}\ \frac{l_{60} - l_{80}}{l_{30}} == \frac{5{,}072{,}326}{9{,}480{,}358} = .5350.$

Example 1.2:

If a mortality table is represented by the function $l_x = 1000\sqrt{100 - x}$

find (i) the probability of a life surviving from birth to age 19

(ii) the probability of a life aged 36 dying before age 51.

Solution:

(i) $_{19}p_o \frac{l_{19}}{l_0}\ \frac{9}{10} = \frac{1{,}000\ \sqrt{100\text{-}19}}{1{,}000\sqrt{100}} = = .9$

(ii) $_{15}q_{36} = \frac{l_{36} - l_{53}}{l_{36}} = \frac{1{,}000\sqrt{64} - 1{,}000\sqrt{49}}{1{,}000\sqrt{64}}\ \frac{1}{8} = = .125.$

Example 1.3:

Complete Table 1.4a.

Table 1.4$_a$

x q_x	l_x	d_x
90	$^1/_3$	3.000
91	$^2/_3$	
92	$^1/_2$	
93	$^2/_3$	
94	$^4/_5$	
95	1	

Solution:

$d_{90} = q_{90} . l_{90} = {}^1/_3 (3,000) = 1,000$

$\therefore l_{91} = 2,000$

$d_{91} = q_{91} . l_{91} = 2/3(2,000) = 800$

$\therefore l_{92} = 1,200$

and proceeding similarly we obtain Table 1.4b

Table 1.4b

x	q_x	l_x	d_x
90	$1/3$	3.000	1,000
91	$2/3$	2.000	800
92	$1/2$	1,200	600
93	$2/3$	600	400
94	$4/5$	200	160
95	1	40	40
96		0	

Example 1.4:

Prove that $_{m|n}q_x = {}_mp_x - {}_{m+n}p_x$

Solution:

$$_{m|n}q_x = \frac{l_{x+m} - l_{x+m+n}}{l_x}$$

$$\frac{l_{x+m} - l_{x+m+n}}{l_x} =$$

$$\frac{l_{x+m}}{l_x} - \frac{l_{x+m+n}}{l_x} =$$

$$= {}_mp_x - {}_{m+n}p_x.$$

Example 1.5:

A life aged 50 is subject to the mortality of the English Life Table No.12-Males, for 10 years, then the a (55) Male table, for 20 years, then the A 1967-70 table with a deduction of 2 years for the remainder of life. Find the probability that the life lives to age 90.

Solution :

In this type of question the probabilities in the age ranges should be kept separate so that l_x **figures on one mortality table are always divided by a figure from the same table. Thus the required probability**

={probability of living} . {probability of living}
{ for 10 years } {for further 20 years}
. {probability of living}
{for further 10 years}

=($_{10}p_{50}$ on ELT No.12-Males).($_{20}p_{60}$ on a (55)-Males)
. ($_{10}p_{78}$ on A 1967-70)

$$= \frac{78,924}{90,085} . \frac{363,991}{859,916} . \frac{4012,83}{14965,5} = .0994.$$

Note. The figures are taken from the l_x column of the tables.

Exercise 1.1:

Using the A 1967- 70 table, find the values of $_{20}p_{40}$, $_{30}q_{25}$, $_{20|15}q_{42}$.

Exercise 1.2:

Using the same mortality as stated in example 1.5 above, find the probability that the life aged 50 dies between the ages of 70 and 85.

Exercise 1.3:

A man aged 56 is subject to the mortality of the English Life Table No. 12- Males, and his brother aged 60 is subject to the mortality of the a (55), Males table. Who is more likely to survive 10 years?

The graph of l_x

So far consideration has been given mainly to exact integral ages- this being the method by which tables are usually tabulated. If the interpretation of the mortality tables as being based on a certain number of lives at an earlier age is considered it might be thought that l_x would be a discontinuous function decreasing by 1 each time one of the lives died during the year. However, this "number alive" interpretation is only a useful shorthand- the l_x figures are as previously discussed only meaningful when divided by each other to give probabilities, and

considered in this way it will be obvious that l_x is a continuous function of x, as has already been mentioned in section 1.2.

Actual determination of l_x at fractional ages when required will mainly be done by some form of interpolation except perhaps where the mortality table is known to have been based on a mathematical expression. The graph of l_x according to the English Life Table No 12- Males is shown in figure 1.1. The general shape of the curve should be noted as well as the obvious fact that l_x is always decreasing. In most tables l_x falls steeply at the infantile ages, and there are at least two points of inflexion.

In the symbol (x) for a life aged exactly x, x may thus not necessarily be an integer.

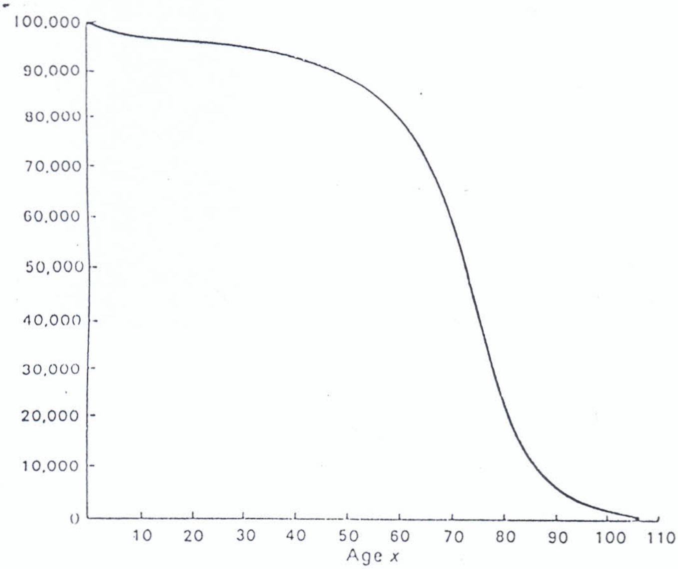


FIGURE 1.1 Graph of l_x—English Life Table No. 12—Males.

7/3
The Force of Mortality

As l_x is a continuous function it can be differentiated and the ratio that the rate of decrease of l_x at age x bears to the value of l_x at that age is called the force of mortality at age x and is represented by the symbol μ_x.

$$\mu_x = -\frac{1}{l_x}\frac{dl_x}{dx} \quad \text{or} \tag{1.6.1}$$

$$= -\frac{d}{dx}\log_e l_x. \tag{1.6.2}$$

This is a measure of the mortality at the precise moment of attaining age x expressed in the form of an annual rate. The negative sign is introduced so that μ_x will be positive, l_x being a decreasing function. The differential coefficient is divided by l_x because as we have seen the absolute magnitude of the l_x figure is arbitrary and depends on the radix of the table, and by dividing by l_x we obtain a figure which is independent of the radix.

Rewriting formula (1.6.1)

$$\frac{dl_x}{dx} = -l_x\mu_x \tag{1.6.3}$$

and integrating over the range α to the limiting age of the table, x being written as $\alpha+t$

$$\int_0^{\omega-\alpha} \frac{dl_{\alpha+t}}{dt}\,dt = -\int_0^{\omega-\alpha} l_{\alpha+t}\mu_{\alpha+t}\,dt$$

i.e.

$$[l_{\alpha+t}]_0^{\omega-\alpha} = -\int_0^{\omega-\alpha} l_{\alpha+t}\mu_{\alpha+t}\,dt$$

$$\therefore\ l_\omega - l_\alpha = -\int_0^{\omega-\alpha} l_{\alpha+t}\mu_{\alpha+t}\,dt$$

and as $l_\omega = 0$

$$l_\alpha = \int_0^{\omega-\alpha} l_{\alpha+t}\mu_{\alpha+t}\,dt.$$

As $l_x = 0$ for all $x > \omega$, ∞ is often substituted for $\omega - \alpha$ in such integrals, and writing x for α we obtain

$$l_x = \int_0^\infty l_{x+t}\mu_{x+t}dt. \qquad (1.6.4)$$

If the limits of integration above had been taken as 0 and 1 we would have obtained

$$[l_{\alpha+t}]_0^1 = -\int_0^1 l_{\alpha+t}\mu_{\alpha+t}dt$$

$$\therefore \; l_{\alpha+1} - l_\alpha = -\int_0^1 l_{\alpha+t}\mu_{\alpha+t}dt$$

and as $l_{\alpha+1} - l_\alpha = -d_\alpha$, we obtain (writing x for α)

$$d_x = \int_0^1 l_{x+t}\mu_{x+t}dt. \qquad (1.6.5)$$

Integrating formula (1.6.2) from age α to $\alpha+n$ again writing $\alpha+t$ for x,

$$\int_0^n \mu_{\alpha+t}dt = -\int_0^n \frac{d}{dt}\log_e l_{\alpha+t}dt$$

$$= -[\log_e l_{\alpha+t}]_0^n$$

$$= -(\log_e l_{\alpha+n} - \log_e l_\alpha)$$

$$= -\log_e \frac{l_{\alpha+n}}{l_\alpha},$$

i.e.

$$\log_e \frac{l_{\alpha+n}}{l_\alpha} = -\int_0^n \mu_{\alpha+t}dt$$

$$\therefore \; \frac{l_{\alpha+n}}{l_\alpha} = e^{-\int_0^n \mu_{\alpha+t}dt}$$

and writing x instead of α

$$_np_x = e^{-\int_0^n \mu_{x+t}dt} \qquad (1.6.6)$$

and

$$l_x = l_0 e^{-\int_0^x \mu_t dt}. \qquad (1.6.7)$$

Formula (1.6.6) bears an obvious and important similarity to the compound interest formula:

Present value of 1 due n years hence $= e^{-\int_0^n \delta_t dt}$, where δ_t is the varying force of interest; thus the force of mortality is analogous to the force of interest.

The probability ${}_nq_x$ can then be expressed

$$ {}_nq_x = 1 - e^{-\int_0^n \mu_{x+t} dt}. \tag{1.6.8}$$

An alternative formula can be derived by integrating formula (1.6.3) from age α to $\alpha+n$ when x is rewritten $\alpha+t$

$$\therefore \ [l_{\alpha+t}]_0^n = -\int_0^n l_{\alpha+t}\mu_{\alpha+t}dt$$

$$\therefore \ l_{\alpha+n} - l_\alpha = -\int_0^n l_{\alpha+t}\mu_{\alpha+t}dt$$

and rewriting x for α and dividing by l_x

$$ {}_nq_x = \frac{l_x - l_{x+n}}{l_x} = \frac{1}{l_x}\int_0^n l_{x+t}\mu_{x+t}dt \tag{1.6.9}$$

or

$$ {}_nq_x = \int_0^n {}_tp_x\mu_{x+t}dt. \tag{1.6.10}$$

When $n = 1$,

$$ q_x = \int_0^1 {}_tp_x\mu_{x+t}dt. \tag{1.6.11}$$

This form of expression is particularly useful when extended to multiple life functions, and can be roughly described as 'keep the life alive until duration t; kill him off at that instant; then add (i.e. integrate) these probabilities over the required time range'.

Thus the probability ${}_{n|m}q_x$ may similarly be expressed

$$ {}_{n|m}q_x = \int_n^{n+m} {}_tp_x\mu_{x+t}dt.$$

The graph of the function $\mu_x l_x$ is called the curve of deaths. An example is given in Figure 1.2 using the same table of mortality as

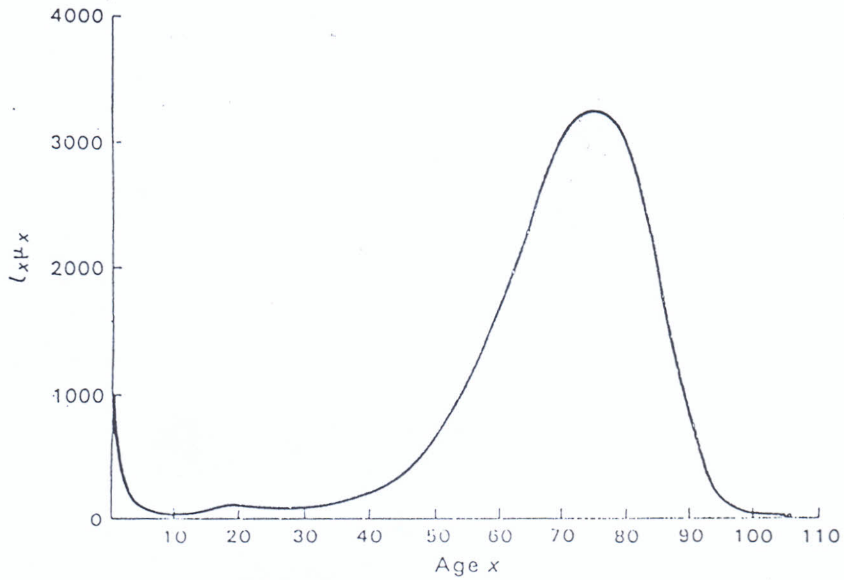


FIGURE 1.2. Graph of $l_x\mu_x$—English Life Table No. 12—Males.

was used to demonstrate the graph of l_x. The maximum and minimum points of the curve will correspond with the points of inflexion on the curve of l_x, as $\frac{dl_x}{dx} = -l_x\mu_x$ so that $\frac{d^2l_x}{dx^2} = 0$ when $\frac{d(l_x\mu_x)}{dx} = 0$.

It will be found from an examination of mortality tables that at some parts of the table $q_x > \mu_x$ and at others $q_x < \mu_x$. From formula (1.6.9)

$$l_xq_x = \int_0^1 l_{x+t}\mu_{x+t}dt.$$

If $l_{x+t}\mu_{x+t}$ is increasing, l_xq_x will exceed the value at the start of the interval i.e. $l_x\mu_x$; so that $q_x > \mu_x$ if the graph of $l_x\mu_x$ is increasing. Considering the graph of l_x, as $-\frac{dl_x}{dx} = l_x\mu_x$ it follows that if $l_x\mu_x$ is increasing $\frac{dl_x}{dx}$ is decreasing, that is the gradient of the tangent to the curve is decreasing and the curve of l_x is concave to the x-axis as demonstrated in Figure 1.3.

So if the curve of l_x is concave to the x-axis $q_x > \mu_x$, and if l_x is convex to the x-axis $q_x < \mu_x$.

Thus μ_x and q_x will be most nearly equal when l_x is most nearly a linear function of x, i.e. near the point of inflexion and will differ most widely where the curvature of l_x is greatest.

Now $\dfrac{dl_x}{dx}$ may be expressed as $\lim\limits_{h\to 0} \dfrac{l_{x+h} - l_x}{h}$, so

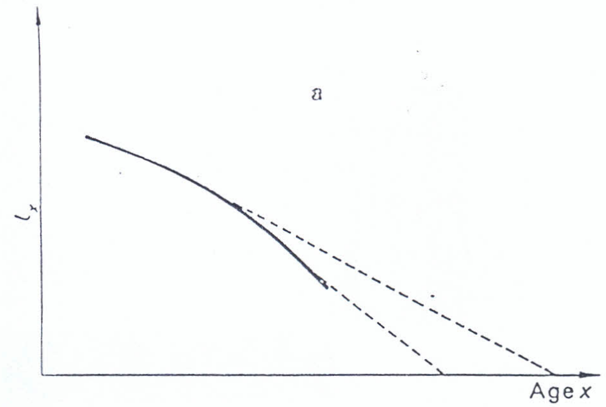
$$\mu_x = -\frac{1}{l_x} \lim_{h\to 0} \frac{l_{x+h} - l_x}{h} = \lim_{h\to 0} \frac{l_x - l_{x+h}}{h\,l_x} = \lim_{h\to 0} \frac{{}_hq_x}{h}$$

and as $\dfrac{{}_hq_x}{h}$ may be regarded as the annual rate of mortality based upon the mortality in the interval x to $x+h$, we see more clearly the statement at the beginning of this section describing μ_x as a nominal annual rate of mortality based on the mortality at the instant of attaining age x.

While by definition q_x cannot exceed 1, as μ_x is an instantaneous rate it may exceed 1 and normally will do so at the beginning and end of a mortality table. Mortality is high in the period immediately following birth and for example in the first hour after birth $\dfrac{1}{(24)(365)}\, q_0$ will considerably exceed $\dfrac{1}{(24)(365)}$ so that the ratio $\dfrac{{}_hq_0}{h}\left(\text{where } h = \dfrac{1}{(24)(365)} \text{ or 1 hour}\right)$ approximating to μ_x will be >1.

In the last year of the mortality table the rate of mortality is 1 for all $x > \omega - 1$, that is $\omega - x < 1$,

$$\therefore \frac{{}_{\omega-x}q_x}{\omega - x} = \frac{1}{\omega - x} > 1, \quad \text{so that } \mu_x > 1.$$



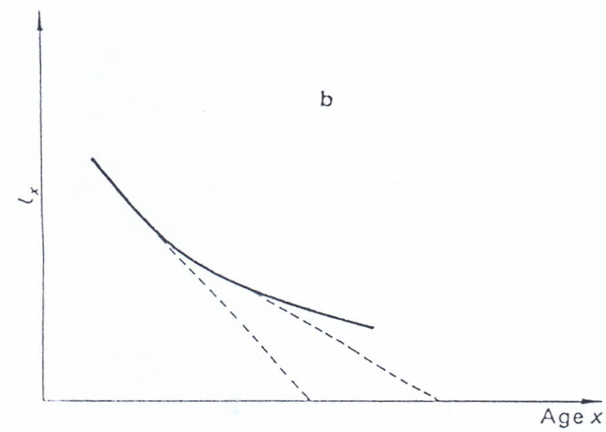


FIGURE 1.3. (*a*) Graph of l_x concave to x-axis. (*b*) Graph of l_x convex to x-axis.

Example 1.6

Find l_x if $\mu_x = \dfrac{1}{100-x}$.

Solution

Using formula (1.6.7) $l_x = l_0 e^{-\int_0^x \mu_t dt}$, where l_0 is the arbitrary radix.

$$\therefore\ l_x = l_0 . e^{-\int_0^x \frac{1}{100-t}dt}$$
$$= l_0 . e^{[\log_e(100-t)]_0^x}$$
$$= l_0 . e^{\log_e(100-x)-\log_e 100}$$
$$= l_0 . e^{\log_e \frac{100-x}{100}}$$
$$= l_0 . \frac{100-x}{100}$$
$$= k(100-x), \quad \text{where } k \text{ is arbitrary.}$$

Example 1.7

Find the differential coefficient of ${}_tp_x$ with respect to t and x.

Solution

$${}_tp_x = \frac{l_{x+t}}{l_x}$$

$$\therefore\ \log_e({}_tp_x) = \log_e l_{x+t} - \log_e l_x,$$

and differentiating with respect to t

$$\frac{1}{{}_tp_x}\frac{d}{dt}({}_tp_x) = \frac{1}{l_{x+t}} . \frac{d}{dt}(l_{x+t}) = -\mu_{x+t}$$

$$\therefore\ \frac{d}{dt}({}_tp_x) = -{}_tp_x . \mu_{x+t}. \qquad (1.6.12)$$

Differentiating with respect to x instead of t

$$\frac{1}{{}_tp_x}\frac{d}{dx}({}_tp_x) = \frac{1}{l_{x+t}}\frac{d}{dx}(l_{x+t}) - \frac{1}{l_x}\frac{d}{dx}(l_x) = -\mu_{x+t} + \mu_x$$

$$\therefore\ \frac{d}{dx}({}_tp_x) = {}_tp_x(\mu_x - \mu_{x+t}). \qquad (1.6.13)$$

These two formulae are worth remembering.

Example 1.8

A life is subject to a constant force of mortality of ·039221. Find the probability

(*a*) That he will live 10 years

(*b*) That he will die within 15 years.

Solution

(*a*) $_{10}p_x = e^{-\int_0^{10}(.039221)dt}$, and as ·039221 is δ at 4% interest

$= v^{10}$ at 4%

$= \cdot 6756$

(*b*) $_{15}p_x$ similarly is v^{15} at 4% = ·5553

$\therefore\ _{15}q_x = 1 - \cdot 5553 = \cdot 4447.$

Exercise 1.4

Show by integrating formula (1.6.10) using the result (1.6.12) that $_nq_x = 1 - {_np_x}$.

Exercise 1.5

A life aged 50 is subject to a constant force of mortality of ·048790. Find the probability that he will die between the ages of 70 and 80.

1.7 Estimation of μ_x from the mortality table

When l_x is tabulated and any underlying mathematical formula is not known, values of μ_x can be found only approximately, the formulae demonstrated below being the most useful.

From formula (1.6.6) $\qquad p_x = e^{-\int_0^1 \mu_{x+t}dt}$

$$\therefore \int_0^1 \mu_{x+t}dt = -\log_e p_x, \tag{1.7.1}$$

sometimes expressed as

$$= \text{colog}_e\, p_x,$$

where 'colog' is written for '−log'. The integral represents the mean value of μ_{x+t} between x and $x+1$ and if it is assumed that this approximates to $\mu_{x+\frac{1}{2}}$ we have

$$\mu_{x+\frac{1}{2}} \doteqdot -\log_e p_x \tag{1.7.2}$$

the approximation being closest where μ_{x+t} does not differ much from a linear function in the range.

Alternatively taking the value of $_2p_{x-1}$,

$$_2p_{x-1} = e^{-\int_{-1}^{1}\mu_{x+t}dt}$$

so that $$\int_{-1}^{1} \mu_{x+t}dt = -\log_e (p_{x-1} \cdot p_x)$$

and if the integral is considered as approximately equal to $2\mu_x$ we get

$$2\mu_x \fallingdotseq -(\log_e p_{x-1} + \log_e p_x)$$

$$\therefore \mu_x \fallingdotseq \tfrac{1}{2}(\operatorname{colog}_e p_{x-1} + \operatorname{colog}_e p_x). \qquad (1.7.3)$$

Another method of approach is to assume that l_x follows a mathematical form, normally a polynomial, and use the Taylor expansion to obtain a value of $\frac{dl_x}{dx}$ (or l'_x) which can be used in the basic formula $\mu_x = -\frac{1}{l_x}\frac{dl_x}{dx}$.

For example, assuming a function of the second degree,

$$l_{x+h} = l_x + hl'_x + \frac{h^2}{2!} l''_x$$

$$\therefore l_{x-1} = l_x - l'_x + \tfrac{1}{2}l''_x$$

and $$l_{x+1} = l_x + l'_x + \tfrac{1}{2}l''_x.$$

Subtracting $$l_{x-1} - l_{x+1} = -2l'_x$$

$$\therefore \mu_x = -\frac{1}{l_x} . l'_x = \frac{1}{2l_x}(l_{x-1} - l_{x+1})$$

$$= \frac{1}{2l_x}(d_{x-1} + d_x). \qquad (1.7.4)$$

A further method is to use the relationship between the differential and difference operator—that is

$$D = \log(1+\Delta) = \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \ldots$$

so that $$\mu_x = -\frac{1}{l_x} Dl_x = -\frac{1}{l_x}(\Delta l_x - \tfrac{1}{2}\Delta^2 l_x + \tfrac{1}{3}\Delta^3 l_x \ldots)$$

i.e. $$\mu_x = \frac{1}{l_x}(d_x - \tfrac{1}{2}\Delta d_x + \tfrac{1}{3}\Delta^2 d_x \ldots). \qquad (1.7.5)$$

This formula might be thought to enable the value of μ_0 to be found whereas the other formulae are not applicable l_{-1} having no mean-

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ing. However, in actual experience μ_x varies rapidly as x increases from 0 to 1 and it is rarely possible to find a satisfactory value for μ_0 if the only information available is the values of l_x at integral ages.

Example 1.9

Find an expression for μ_x assuming that l_x is a 4th degree polynomial.

Solution

Using the Taylor expansion

$$l_{x+h} = l_x + hl'_x + \frac{h^2}{2!}l''_x + \frac{h^3}{3!}l'''_x + \frac{h^4}{4!}l^{\text{iv}}_x,$$

$$l_{x-2} = l_x - 2l'_x + 2l''_x - \tfrac{4}{3}l'''_x + \tfrac{2}{3}l^{\text{iv}}_x \qquad (1)$$

$$l_{x-1} = l_x - l'_x + \tfrac{1}{2}l''_x - \tfrac{1}{6}l'''_x + \tfrac{1}{24}l^{\text{iv}}_x \qquad (2)$$

$$l_{x+1} = l_x + l'_x + \tfrac{1}{2}l''_x + \tfrac{1}{6}l'''_x + \tfrac{1}{24}l^{\text{iv}}_x \qquad (3)$$

$$l_{x+2} = l_x + 2l'_x + 2l''_x + \tfrac{4}{3}l'''_x + \tfrac{2}{3}l^{\text{iv}}_x. \qquad (4)$$

We wish to eliminate all the derivatives except l'_x. Subtracting formula (4) from (1), and (3) from (2) gives

$$l_{x-2} - l_{x+2} = -(4l'_x + \tfrac{8}{3}l'''_x) \qquad (5)$$

$$l_{x-1} - l_{x+1} = -(2l'_x + \tfrac{1}{3}l'''_x). \qquad (6)$$

Multiplying formula (6) by 8 and subtracting from (5) gives

$$(l_{x-2} - l_{x+2}) - 8(l_{x-1} - l_{x+1}) = -4l'_x + 16l'_x = 12l'_x$$

$$\therefore \mu_x = -\frac{l'_x}{l_x} \doteqdot -\frac{1}{12l_x}((l_{x-2} - l_{x+2}) - 8(l_{x-1} - l_{x+1}))$$

$$\left.\begin{aligned} &= \frac{1}{12l_x}(8(l_{x-1} - l_{x+1}) - (l_{x-2} - l_{x+2})) \\ \text{or}\quad &\frac{1}{12l_x}(7(d_{x-1} + d_x) - (d_{x-2} + d_{x+1})) \end{aligned}\right\} \qquad (1.7.6)$$

Example 1.10

Find a value for μ_{90} on the A1967–70 table by four different formulae.

Solution

Formula (1.7.3) gives $\mu_{90} \doteqdot -\frac{1}{2}(\log_e p_{89} + \log_e p_{90})$

$$= -\tfrac{1}{2}(\log_e \cdot 79896 + \log_e \cdot 78349)$$
$$= -\tfrac{1}{2}(-\cdot 22443 - \cdot 24400)$$
$$= \cdot 23422.$$

Formula (1.7.4) gives $\mu_{90} \doteqdot \dfrac{1}{2l_{90}}(d_{89} + d_{90})$

$$= \frac{656\cdot 37 + 564\cdot 78}{2(2608\cdot 53)}$$
$$= \cdot 23407.$$

Formula (1.7.5) requires a table of the differences of d_x thus:

x	d_x	Δd_x	$\Delta^2 d_x$	$\Delta^3 d_x$	$\Delta^4 d_x$
90	564·78				
		−88·97			
91	475·81		5·02		
		−83·95		2·06	
92	391·86		7·08		−0·55
		−76·87		1·51	
93	314·99		8·59		
		−68·28			
94	246·71				

$$\mu_{90} \doteqdot \frac{1}{2608\cdot 53}(564\cdot 78 - \tfrac{1}{2}(-88\cdot 97) + \tfrac{1}{3}(5\cdot 02) - \tfrac{1}{4}(2.06) + \tfrac{1}{5}(-0\cdot 55)\ldots)$$
$$= \frac{1}{2608\cdot 53}(564\cdot 78 + 44\cdot 48 + 1\cdot 67 - 0\cdot 52 - 0\cdot 11)$$
$$= \frac{610\cdot 30}{2608\cdot 53}$$
$$= \cdot 23396.$$

Formula (1.7.6) gives $\mu_{90} \doteqdot \dfrac{7(d_{89} + d_{90}) - (d_{88} + d_{91})}{12 l_{90}}$

$$= \frac{7(656\cdot 37 + 564\cdot 78) - (747\cdot 93 + 475\cdot 81)}{12(2608\cdot 53)}$$
$$= \cdot 23399.$$

The tabulated value is $\cdot 23398$.

Exercise 1.6

Use two approximate formulae to find values of μ_{80} according to the $a(55)$ Male table and compare them with the tabulated value.

Exercise 1.7

If $l_x = 100\sqrt{100-x}$ find μ_{84} exactly and by using an approximate method.

7/4
Ultimate and Aggregate Mortality Tables

Select, ultimate and aggregate mortality tables

So far consideration has been given to tables where the rates of mortality depend on the age of a member of the group concerned, the group perhaps being the population of a country, the class of assured lives or the class of annuitant lives. Where attention is paid also to the duration for which a member has been within the group the table is called a 'Select' table. Such a table in the case of a population table might thus take account specially of mortality of immigrant lives, but the more common use is with assured lives and annuitants' tables where it is reasonable to assume that the lives concerned have been 'selected', and are likely to experience lighter mortality than the group as a whole at the same age-the assured lives because of the medical evidence obtained before a life assurance office accepts a proposal, and the annuitant lives because a person who is ill is unlikely to purchase an annuity (this is known as self-selection, being exercised by the lives themselves**). While normally select mortality is lighter than ordinary it is possible for the mortality to be heavier, for example, the mortality of ill-health retrials from pension schemes is likely to be particularly heavy for the first few years after retrial, in which case it may be referred to as reversed or negative selection.**

A mortality table in which no regard is paid to period of membership in the group, such as a population table, is called an 'aggregate' table.

While some mortality investigations have indicated that the effect of selection may continue indefinitely implying that a different table will be required for each entry age it is nowadays common practice to use a short 'select' period of one or two years. The A1967-70 and a (55) tables have select periods of two and one years respectively. The implication of a select period of two years is that one mortality rate is used at each age for all those who have been in the group for at least two years. This part of the table is referred to as an 'ultimate' mortality table and the normal l_x and other symbols are used. For the functions in the select part of a select and ultimate table square brackets are used, thus $q_{(x)}$ is the rate of mortality for a life aged exactly x who has just joined the group. If the select period is longer than one year further symbols are required and

the rate of mortality for a life aged *x+l* who joined the group one year ago is $q_{(x)+1}$, and similarly the rate of mortality for a life aged *x+r* who joined the group *r* years ago is $q_{(x)+r}$ unless the select period is less than *r* years when the brackets can be omitted .

The l_x columns of a select and ultimate table are set out as shown in table 1.5 extracted from the A1967-70 table (taking the figures to one decimal place rather than three as published**). The figures in heavy type show the table as it relates to a life entering at age 52- the figures are considered on the horizontal line until the ultimate part of the table is reached when the figures then continue downwards. Except in the right- hand column of *l's* (**the ultimate column**) figures should never be related in the same vertical column. Tables set out in a similar way are used for the other mortality functions such as $d_{(x)}$,$p_{(x)}$,$q_{(x)}$,$u_{(x)}$, when the select period is one year as in a (55) the values for (x) and (x) may however be put on the same line.**

Table1.5

Age(x)	$l_{(x)}$	$l_{(x)+1}$	l_{x+2}	Age x+2
50	32,558.00	32,464.80	32,338.60	52
51	32,383.80	32,282.00	32,143.50	53
52	32,188.70	32,078.00	31,926.40	54
53	31,970.90	31,850.60	31,685.20	55
54	31,728.20	31,597.90	31,417.70	56
55	31,458.30	31,317.60	31,121.80	57

A graph of some of the values of $q_{(x)}$ and q_x on the A 1967-70 mortality table is given in Figure 1.4, the select values commencing at ten-year intervals.

It will be seen that the large number of select mortality tables can be represented by on compact select- ultimate table by the choice of the radixes $l_{(x)}$ so that the ultimate portion of each of the tables is identical. Thus in constructing the table $l_{(x)}, l_{(x)+1}$, etc., are calculated in the reverse order from the method of section 1.4. If r is the select period of the table then to find the values commencing at $l_{(x)}$, $l_{(x)+r-1}$ would first be found from

$$l_{(x)+r-1} = \frac{l_{x+r}}{p_{(x)+r-1}}$$

and $l_{(x)+r-2}$ from

$$l_{(x)+r-2} = \frac{l_{(x)+r-1}}{p_{(x)+r-2}}$$

and so on. The values of $d_{(x)}$, $d_{(x)+1}$, … would then be found by subtracting the $l_{(x)}, l_{(x)+1}$, etc.

The normal relationships hold between the various mortality functions for a select-ultimate table, the only difference being that the age suffix is expressed in a different way until the ultimate portion of the table is reached. For

example, in the above table with a select period of two years, the mortality rates experienced in the first three years by a life entering the group at age x are

$$q_{(x)} = \frac{d(x)}{l(x)} = \frac{l(x) - l(x)+1}{l(x)}$$

$$q_{(x)+1} = \frac{d(x)+1}{l(x)+1} = \frac{l(x)+1 - lx+2,}{l(x)+1}$$

$$q_{x+2} = \frac{lx+2 - lx+3}{lx+2}$$

For example, for x= 52, the rates of mortality subsequently experienced will be (again reducing the number of significant figures when reading from the tables**)**

$$q_{(52)} = \frac{32{,}188.7 - 32{,}078.0}{32{,}188.7} = .00344,$$

$$q_{(52)+1} = = \frac{32{,}078.0 - 31{,}926.4}{32{,}078.0} \quad .00473$$

$$q_{54} = = \frac{31{,}926.4 - 31{,}685.2}{31{,}926.4} \quad .00756$$

$$q_{55} = = \frac{31{,}685.2 - 31{,}417.7}{31{,}685.2} \quad .00844$$

It will be seen that there are three different rates of mortality for each age, depending on duration, for example, at age 52 the rates of mortality would be if just joined group = $q_{(52)}$ = .00344 (as above**) if joined group one year ago.**

$$= q_{(51)+1} = \frac{l(51)+1 - l53}{l(51)+1}$$

$$= \frac{32{,}282.0 - 32{,}143.5}{32{,}282.0}$$

= .00429.

If joined group two or more years ago

$$= q_{(52)} = \frac{l_{52} - l_{53}}{l_{52}}$$

$$= \frac{32{,}338.6 - 32{,}143.5}{32{,}338.6}$$

= .00603.

It can be seen, as one would expect, that the mortality rates are less the nearer the life is to entry, $q_{(52)}$ being 57% and $q_{(51)+1}$ 71%of q_{52}.

A formula of the type of (1.7.3), (1.7.4) or (1.7.6) cannot be used to find $u_{(x)}$ as it would introduce meaningless symbols such as $d_{(x)-1}$. Even the use of an advancing difference formula such as (1.7.5) may not give a satisfactory answer because the run of values of $d_{(x)}$, etc., cause the formula to converge only very slowly. Also during the select period the graph of $u_{(x)}$ is likely to be of a different shape from the ultimate giving discontinuity in the curve at the end of the select period- this is also apparent from the graph of $q_{(x)}$ and q_x in Figure 1.4.

Example 1.15:

Write in terms of l's an expression for ${}_3l_4q_{(40)+2}$, if the select period of the table is six years.

Solution:

$${}_3l_4q_{(40)+2} = \frac{l_{(40)+5} - l_{49}}{l_{(40)+2}}$$

Example 1.16:

Find the values of the following using the table in the previous section:

(i) ${}_5p_{(50)}$, (ii) ${}_2q_{(51)}$, (iii) ${}_3p_{(51)+1}$, (iv) ${}_1l_3q_{(53)}$

Solution:

(i) ${}_5p_{(50)} = = \frac{31,685.2}{32,558.0} = \frac{l_{55}}{l_{(50)}}$ .9732,

(ii) ${}_2q_{(51)}$ $\frac{l_{(51)} - l_{53}}{l_{(51)}} = = \frac{32,383.8 - 32,143.5}{32,383.8} = .0074$,

(iii) ${}_3p_{(51)+1} = \frac{l_{55}}{l_{(51)+1}} = \frac{31,685.2}{32,282.0} = .9815$,

(iv) ${}_1l_3q_{(53)} = \frac{l_{(53)+1} - l_{57}}{l_{(53)}} = \frac{31,850.6 - 31,121.8}{31,970.9} = .0228$.

Exercise 1.12:

Find the values of the following using the table in the previous section:

(i) ${}_4p_{(50)+1}$, (ii) ${}_3l_2q_{(50)}$ (iii) ${}_4q_{(51)+1}$

Exercise 1.13:

It is required to construct a select table with select period two years to be added to the English Life Table No. 12-Males which is to be treated as the ultimate table. If $q_{\{x\}} = {}^{1}/_{2}\, q_x$ and $q_{\{x\}+1} = {}^{2}/_{3}\, q_{x+1}$, find the value of $l_{\{60\}}$.